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The inverse of any two-by-two nonsingular partitioned matrix and three matrix inverse completion problems

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ABSTRACT

A formula for the inverse of any nonsingular matrix partitioned into two-by-two blocks is derived through a decomposition of the matrix itself and generalized inverses of the submatrices in the matrix. The formula is then applied to three matrix inverse completion problems to obtain their complete solutions.

1. Introduction

Throughout this paper, $\mathbb{C}^{m\times n}$ stands for the set of all $m\times n$ complex matrices. The symbols M^*, $r(M)$, $\mathscr{R}(M)$ and $\mathscr{N}(M)$ stand for the conjugate transpose, rank, range (column space) and null space of a complex matrix $M\in\mathbb{C}^{m\times n}$, respectively; $[A \mid B]$ denotes a row block matrix consisting of A and B. The Moore–Penrose inverse of $M\in\mathbb{C}^{m\times n}$, denoted by $M^\dagger$, is the unique solution to the four matrix equations

$$\text{(i) } MXM=M,\qquad \text{(ii) } XMX=X,\qquad \text{(iii) } (MX)^*=MX,\qquad \text{(iv) } (XM)^*=XM.$$

Further, let $E_M=I_m-MM^\dagger$ and $F_M=I_n-M^\dagger M$ stand for the two orthogonal projectors onto the kernels of M^* and M, respectively. One of the most important applications of Moore–Penrose inverses is to derive closed-form formulas for ranks of partitioned matrices, as well as general solutions of matrix equations; see, e.g., Lemmas 1.1–1.6 below.

Consider a 2×2 partitioned matrix

$$M=\begin{bmatrix} A & B\\ C & D\end{bmatrix}, \tag{1.1}$$

where $A\in\mathbb{C}^{m\times n}$, $B\in\mathbb{C}^{m\times k}$, $C\in\mathbb{C}^{l\times n}$ and $D\in\mathbb{C}^{l\times k}$ are given matrices with $m+l=n+k=t$. If A is square and nonsingular, then M can be decomposed as

$$M=\begin{bmatrix} I_m & 0\\ CA^{-1} & I_l\end{bmatrix}\begin{bmatrix} A & 0\\ 0 & D-CA^{-1}B\end{bmatrix}\begin{bmatrix} I_m & A^{-1}B\\ 0 & I_l\end{bmatrix}. \tag{1.2}$$

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This decomposition is called the Aitken block-diagonalization formula; see, e.g., Puntanen and Styan [1]. Moreover, if both M and A in (1.1) are nonsingular, then the Schur complement $S_A = D - CA^{-1}B$ of A in M is also nonsingular, and the inverse of M can be written in the following form

$$M^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BS_A^{-1}CA^{-1} & -A^{-1}BS_A^{-1} \\ -S_A^{-1}CA^{-1} & S_A^{-1} \end{bmatrix}. \tag{1.3}$$

This is often called in the literature the Banachiewicz inversion formula for the inverse of a nonsingular partitioned matrix; see, e.g., Puntanen and Styan [1]. Some variations of (1.3) can be found in [2]. Eqs. (1.2) and (1.3) are two of the most useful formulas in matrix theory and applications, and were extensively applied to manipulate various operations related to partitioned matrices and their inverses. If, however, none of the four submatrices in (1.1) is nonsingular, then (1.3) no longer holds. In this case, we have to use generalized inverses of the submatrices in M to construct the partitioned expression of the inverse of M. In an earlier paper [3] on the Moore–Penrose inverses of partitioned matrices under rank additivity conditions, the first author of the present paper obtained the following formula for the inverse of M in (1.1):

$$M^{-1} = \begin{bmatrix} H_1 - H_2CA^{\dagger} - A^{\dagger}BH_3 + A^{\dagger}BJ^{\dagger}CA^{\dagger} & H_2 - A^{\dagger}BJ^{\dagger} \\ H_3 - J^{\dagger}CA^{\dagger} & J^{\dagger} \end{bmatrix}, \tag{1.4}$$

where

$$H_1 = A^{\dagger} + C_1^{\dagger}(S_AJ^{\dagger}S_A - S_A)B_1^{\dagger}, \qquad H_2 = C_1^{\dagger}(I - S_AJ^{\dagger}), \qquad H_3 = (I - J^{\dagger}S_A)B_1^{\dagger},$$
$$S_A = D - CA^{\dagger}B, \qquad B_1 = E_AB, \qquad C_1 = CF_A, \qquad J = E_{C_1}S_AF_{B_1}.$$

In Section 2, we give a direct proof of (1.4) and present some consequences.

A partial (incomplete) matrix is a matrix whose entries are specified only for a subset of positions in the matrix. A matrix completion problem refers to the choice of the unspecified entries of the partial matrix such that the resultant matrix has certain prescribed properties on its determinant, rank, range, null space, inverse, norm, eigenvalues, characteristic polynomial, singular values, definiteness, idempotency, orthogonality, etc. Many results on completions of partial (operator) matrices and their applications can be found in the literature; see [4–24] among others. A well-known matrix completion problem is to assign values to the unspecified entries so as to maximize or minimize the resulting matrix rank. This problem has deep connections with computational complexity and numerous important algorithmic applications. Determining the complexity of this problem is a fundamental open question in computational complexity. Under different settings of unknown entries in a partial matrix, the problem is now known as in P, in RP, or NP-hard; see [11,14,18,25] among others.

One of the simplest partial matrices associated with M in (1.1) is given by

$$M(X) = \begin{bmatrix} A & B \\ C & X \end{bmatrix}, \tag{1.5}$$

where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$ are given with $m + l = n + k = t$, and $X \in \mathbb{C}^{l\times k}$ is a variable matrix. It is obvious that the matrix $M(X)$ varies with the choice of the variable matrix X. Hence it is possible to choose the matrix X such that the resulting $M(X)$ has certain prescribed properties. In this paper, we reconsider the following three completion problems related to the nonsingularity and the inverse of $M(X)$ in (1.5):

(i) Find a matrix X such that the matrix $M(X)$ in (1.5) is nonsingular and its inverse has the form

$$M^{-1}(X) = \begin{bmatrix} ? & ? \\ ? & G \end{bmatrix}, \tag{1.6}$$

where $G \in \mathbb{C}^{k\times l}$ is a given matrix.

(ii) Let

$$M_1(X) = \begin{bmatrix} A & B \\ B^* & X \end{bmatrix}, \tag{1.7}$$

where $A = A^* \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{m\times n}$ are given. Find $X = X^* \in \mathbb{C}^{n\times n}$ such that $M_1(X)$ is nonsingular and its inverse has the form

$$M_1^{-1}(X) = \begin{bmatrix} ? & ? \\ ? & G \end{bmatrix}, \tag{1.8}$$

where $G = G^* \in \mathbb{C}^{n\times n}$ is given.

(iii) Let

$$M_2(X) = \begin{bmatrix} A & B \\ -B^* & X \end{bmatrix}, \tag{1.9}$$

where $A = -A^* \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{m\times n}$ are given. Find $X = -X^* \in \mathbb{C}^{n\times n}$ such that $M_2(X)$ is nonsingular and its inverse has the form

$$M_2^{-1}(X) = \begin{bmatrix} ? & ? \\ ? & G \end{bmatrix}, \tag{1.10}$$

where $G = -G^* \in \mathbb{C}^{n\times n}$ is given.

The problems outlined in (1.6), (1.8) and (1.10) were studied in [8,10]. In Section 3, we give complete solutions to these three problems through (1.4).

Some well-known rank formulas for partitioned matrices due to Marsaglia and Styan [26] are given below.

Lemma 1.1. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$ and $D \in \mathbb{C}^{l\times k}$, respectively. Then*

$$r[A \mid B] = r(A) + r(E_AB), \tag{1.11}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A), \tag{1.12}$$

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r\begin{bmatrix} 0 & E_AB \\ CF_A & S_A \end{bmatrix}, \tag{1.13}$$

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} + r[A \mid B] - r(A) + r(E_{C_1}S_AF_{B_1}), \tag{1.14}$$

where $S_A = D - CA^\dagger B$, $B_1 = E_AB$ and $C_1 = CF_A$.

Lemma 1.2 ([27]). *Suppose A, B and C satisfy the following rank additivity condition*

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r\begin{bmatrix} A \\ C \end{bmatrix} + r(B) = r[A \mid B] + r(C). \tag{1.15}$$

Then

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^\dagger = \begin{bmatrix} L^\dagger & C^\dagger - L^\dagger AC^\dagger \\ B^\dagger - B^\dagger AL^\dagger & B^\dagger(AL^\dagger A - A)C^\dagger \end{bmatrix}, \tag{1.16}$$

where $L = E_BAF_C$.

The following result is derived from (1.14).

Lemma 1.3. *Let $M(X)$ be as given in (1.5). Then,*

(a) *The maximal and the minimal ranks of $M(X)$ with respect to $X \in \mathbb{C}^{l\times k}$ are given by the following two explicit formulas*

$$\max_{X\in\mathbb{C}^{l\times k}} r[M(X)] = \min\left\{ r\begin{bmatrix} A \\ C \end{bmatrix} + k,\ r[A \mid B] + l \right\}, \tag{1.17}$$

$$\min_{X\in\mathbb{C}^{l\times k}} r[M(X)] = r\begin{bmatrix} A \\ C \end{bmatrix} + r[A \mid B] - r(A). \tag{1.18}$$

(b) *There exists a matrix X such that $M(X)$ is nonsingular if and only if*

$$r\begin{bmatrix} A \\ C \end{bmatrix} = n \quad \text{and} \quad r[A \mid B] = m$$

hold. In this case, the matrix X is determined by the rank equation

$$r[E_{C_1}(X - CA^\dagger B)F_{B_1}] = t - m - n + r(A),$$

where $B_1 = E_AB$ and $C_1 = CF_A$.

(c) *The matrix $M(X)$ is nonsingular for any X if and only if*

$$r(A) = m + n - t, \qquad r[A \mid B] = m, \qquad r\begin{bmatrix} A \\ C \end{bmatrix} = n,$$

or equivalently,

$$\mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}, \qquad \mathscr{R}(A^*) \cap \mathscr{R}(C^*) = \{0\},$$
$$r(A) = m + n - t, \qquad r(B) = k, \qquad r(C) = l.$$

Proof. Applying (1.14) to $M(X)$ gives

$$r[M(X)] = r\begin{bmatrix} A \\ C \end{bmatrix} + r[A \mid B] - r(A) + r[E_{C_1}(X - CA^{\dagger}B)F_{B_1}]. \tag{1.19}$$

It is obvious that

$$\max_{X\in\mathbb{C}^{l\times k}} r[E_{C_1}(X - CA^{\dagger}B)F_{B_1}] = \max_{Z\in\mathbb{C}^{l\times k}} r(E_{C_1}ZF_{B_1}) = \min\{r(E_{C_1}), r(F_{B_1})\}. \tag{1.20}$$

Applying (1.11) and (1.12) also gives

$$r(E_{C_1}) = l + r(A) - r\begin{bmatrix} A \\ C \end{bmatrix} \quad \text{and} \quad r(F_{B_1}) = k + r(A) - r[A \mid B]. \tag{1.21}$$

Combining (1.20) and (1.21) gives

$$\max_{X} r[E_{C_1}(X - CA^{\dagger}B)F_{B_1}] = \min\left\{ l + r(A) - r\begin{bmatrix} A \\ C \end{bmatrix}, k + r(A) - r[A \mid B] \right\}. \tag{1.22}$$

It is also easy to see that

$$\min_{X\in\mathbb{C}^{l\times k}} r[E_{C_1}(X - CA^{\dagger}B)F_{B_1}] = \min_{Z\in\mathbb{C}^{l\times k}} r(E_{C_1}ZF_{B_1}) = 0. \tag{1.23}$$

Combining (1.22) and (1.23) with (1.19) yields (1.17) and (1.18). The results in (b) and (c) follow from (1.17) and (1.18). □

Lemma 1.4 ([28]). *Let $A \in \mathbb{C}^{m\times p}$, $B \in \mathbb{C}^{q\times n}$ and $C \in \mathbb{C}^{m\times n}$ be given. Then the matrix equation $AXB = C$ is solvable for X if and only if $\mathscr{R}(C) \subseteq \mathscr{R}(A)$ and $\mathscr{R}(C^*) \subseteq \mathscr{R}(B^*)$, or equivalently, $AA^{\dagger}CB^{\dagger}B = C$. In this case, the general solution can be written in the following parametric form*

$$X = A^{\dagger}CB^{\dagger} + F_AU_1 + U_2E_B,$$

where $U_1, U_2 \in \mathbb{C}^{p\times q}$ are arbitrary.

Lemma 1.5 ([29]). *Let $A \in \mathbb{C}^{m\times n}$ and $B = B^* \in \mathbb{C}^{n\times n}$ be given. Then the matrix equation $A^*XA = B$ has an Hermitian solution X if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A^*)$. In this case, the general Hermitian solution can be written in the following parametric form*

$$X = (A^*)^{\dagger}BA^{\dagger} + E_AW^* + WE_A,$$

where $W \in \mathbb{C}^{m\times m}$ is arbitrary.

An analogous result on the skew-Hermitian solution to $A^*XA = B$ is given below.

Lemma 1.6. *Let $A \in \mathbb{C}^{m\times n}$ and $B = -B^* \in \mathbb{C}^{n\times n}$ be given. Then the matrix equation $A^*XA = B$ has a skew-Hermitian solution for X if and only if $\mathscr{R}(B) \subseteq \mathscr{R}(A^*)$. In this case, the general skew-Hermitian solution can be written in the following parametric form*

$$X = (A^*)^{\dagger}BA^{\dagger} - E_AW^* + WE_A,$$

where $W \in \mathbb{C}^{m\times m}$ is arbitrary.

2. The inverse of a 2 × 2 nonsingular partitioned matrix

In this section, we give a direct proof for (1.4), and then present some consequences.

Theorem 2.1. *Suppose the matrix M in (1.1) is nonsingular. Then the inverse of M can be expressed as (1.4).*

Proof. It is easy to verify that the matrix M in (1.1) can be decomposed as

$$M = \begin{bmatrix} I_m & 0 \\ CA^{\dagger} & I_l \end{bmatrix}\begin{bmatrix} A & B_1 \\ C_1 & S_A \end{bmatrix}\begin{bmatrix} I_n & A^{\dagger}B \\ 0 & I_k \end{bmatrix} := PNQ, \tag{2.1}$$

where $S_A = D - CA^{\dagger}B$, $B_1 = E_AB$ and $C_1 = CF_A$. In this case, if M is nonsingular, then the inverse of M can be expressed as

$$M^{-1} = Q^{-1}N^{-1}P^{-1} = \begin{bmatrix} I_n & -A^{\dagger}B \\ 0 & I_k \end{bmatrix}\begin{bmatrix} A & B_1 \\ C_1 & S_A \end{bmatrix}^{-1}\begin{bmatrix} I_m & 0 \\ -CA^{\dagger} & I_l \end{bmatrix}. \tag{2.2}$$

In order to find N^{-1} in (2.2), we decompose the matrix N in (2.1) as

$$N = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B_1 \\ C_1 & S_A \end{bmatrix} := N_1 + N_2.$$

Observe that $N_1^* N_2 = 0$ and $N_2 N_1^* = 0$. Hence it is easy to verify by definition that

$$N^{-1} = N_1^\dagger + N_2^\dagger \tag{2.3}$$

holds. Since the matrix N in (2.1) is nonsingular, its rank satisfies

$$r(N) = r\begin{bmatrix} A \\ C_1 \end{bmatrix} + r\begin{bmatrix} B_1 \\ S_A \end{bmatrix} = r(A) + r(C_1) + r\begin{bmatrix} B_1 \\ S_A \end{bmatrix}, \tag{2.4}$$

$$r(N) = r[A \mid B_1] + r[C_1 \mid S_A] = r(A) + r(B_1) + r[C_1 \mid S_A]. \tag{2.5}$$

Combining (2.4) and (2.5) with (1.11), (1.12) and (1.13) gives

$$r(N_2) = r\begin{bmatrix} 0 & B_1 \\ C_1 & S_A \end{bmatrix} = r(C_1) + r\begin{bmatrix} B_1 \\ S_A \end{bmatrix} = r(B_1) + r[C_1 \mid S_A].$$

In this case, applying Lemma 1.2 to N_2 gives

$$N_2^\dagger = \begin{bmatrix} C_1^\dagger (S_A J^\dagger S_A - S_A) B_1^\dagger & C_1^\dagger - C_1^\dagger S_A J^\dagger \\ B_1^\dagger - J^\dagger S_A B_1^\dagger & J^\dagger \end{bmatrix}. \tag{2.6}$$

Substituting (2.6) into (2.3) and then (2.3) into (2.2) gives

$$M^{-1} = Q^{-1}\begin{bmatrix} A^\dagger + C_1^\dagger (S_A J^\dagger S_A - S_A) B_1^\dagger & C_1^\dagger - C_1^\dagger S_A J^\dagger \\ B_1^\dagger - J^\dagger S_A B_1^\dagger & J^\dagger \end{bmatrix} P^{-1}. \tag{2.7}$$

Written in a partitioned form, (2.7) becomes (1.4). □

In addition to (2.1), the matrix M in (1.1) can also be decomposed as other three products involving the Schur complements $S_B = C - DB^\dagger A$, $S_C = B - AC^\dagger D$ and $S_D = A - BD^\dagger C$, respectively. In these cases, three new formulas for the inverse of M can also be derived from the decompositions. All these formulas can be used to study various problems related to inverses of partitioned matrices. For instance, comparing the submatrices in the four inversion formulas will yield a variety of equalities for the Moore–Penrose inverses of submatrices in M; applying these inversion formulas to any $s \times t$ nonsingular partitioned matrix will give various partitioned expressions for the inverse of the matrix; applying (1.4) to (1.6), (1.8) and (1.10) will give solutions that satisfy the three equations.

If the submatrices A, B, C and D in (1.1) satisfy some additional conditions, then (1.4) may reduce to various simpler forms. For example, if A^{-1} exists, (1.4) reduces to (1.3). Another special case of (1.4) is given below, which will be used in Section 3.

Corollary 2.2. *Suppose that the matrix M in (1.1) is nonsingular and the submatrices in M satisfy the following four conditions*

$$\mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}, \qquad \mathscr{R}(A^*) \cap \mathscr{R}(C^*) = \{0\}, \qquad \mathscr{R}(D) \subseteq \mathscr{R}(C), \qquad \mathscr{R}(D^*) \subseteq \mathscr{R}(B^*). \tag{2.8}$$

Then the inverse of M can be expressed as

$$M^{-1} = \begin{bmatrix} A^\dagger - A^\dagger B B_1^\dagger - C_1^\dagger C A^\dagger - C_1^\dagger S_A B_1^\dagger & C_1^\dagger \\ B_1^\dagger & 0 \end{bmatrix}, \tag{2.9}$$

where $S_A = D - CA^\dagger B$, $B_1 = E_A B$ and $C_1 = CF_A$.

Proof. Under the conditions in (2.8), it is easy to verify that

$$B_1 B_1^\dagger = BB^\dagger, \qquad C_1^\dagger C_1 = C^\dagger C, \qquad CC^\dagger D = D, \qquad DB^\dagger B = D.$$

Hence it follows that $J = E_{C_1} S_A F_{B_1} = E_C (D - CA^\dagger B) F_B = 0$. Thus (1.4) reduces to (2.9). □

The following result on the inverse of a bordered matrix follows from Theorem 2.1 and Lemma 1.2.

Corollary 2.3. *Suppose that $M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ is nonsingular. Then M^{-1} can be expressed in the following two forms*

$$\begin{aligned} M^{-1} &= \begin{bmatrix} H_1 - H_2 C A^\dagger - A^\dagger B H_3 + A^\dagger B J^\dagger C A^\dagger & H_2 - A^\dagger B J^\dagger \\ H_3 - J^\dagger C A^\dagger & J^\dagger \end{bmatrix} \\ &= \begin{bmatrix} L^\dagger & C^\dagger - L^\dagger A C^\dagger \\ B^\dagger - B^\dagger A L^\dagger & B^\dagger (A L^\dagger A - A) C^\dagger \end{bmatrix}, \end{aligned}$$

where

$$S_A = -CA^\dagger B, \qquad B_1 = E_A B, \qquad C_1 = CF_A, \qquad J = E_{C_1} S_A F_{B_1},$$
$$H_1 = A^\dagger + C_1^\dagger (S_A J^\dagger S_A - S_A) B_1^\dagger, \qquad H_2 = C_1^\dagger (I - S_A J^\dagger), \qquad H_3 = (I - J^\dagger S_A) B_1^\dagger, \qquad L = E_B A F_C.$$

3. Completing partitioned matrices and their inverses

As direct applications of the inversion formula in (1.4), we are now able to completely solve the matrix inverse completion problems corresponding to (1.6), (1.8) and (1.10).

Theorem 3.1 ([10]). *Let $M(X)$ be as given in (1.5). Then there exists a matrix X such that (1.6) holds if and only if A, B, C and G satisfy the following five conditions*

$$r\begin{bmatrix} A \\ C \end{bmatrix} = n, \qquad r[A \mid B] = m, \tag{3.1}$$
$$r(G) = t - m - n + r(A), \tag{3.2}$$
$$\mathscr{R}(BG) \subseteq \mathscr{R}(A), \qquad \mathscr{R}[(GC)^*] \subseteq \mathscr{R}(A^*). \tag{3.3}$$

Proof. It can be seen from (1.4) and Lemma 1.3(b) and (c) that there exists a matrix X such that (1.6) holds if and only if A, B and C satisfy (3.1), and the following two equalities

$$r[E_{C_1}(X - CA^\dagger B)F_{B_1}] = t - m - n + r(A) \quad \text{and} \quad [E_{C_1}(X - CA^\dagger B)F_{B_1}]^\dagger = G$$

hold. These two equalities are also equivalent to

$$r(G) = t - m - n + r(A) \quad \text{and} \quad E_{C_1}(X - CA^\dagger B)F_{B_1} = G^\dagger. \tag{3.4}$$

From Lemma 1.4, the second equation in (3.4) is solvable for X if and only if

$$\mathscr{R}(G^\dagger) \subseteq \mathscr{R}(E_{C_1}) = \mathscr{N}(C_1^*) \quad \text{and} \quad \mathscr{R}[(G^\dagger)^*] \subseteq \mathscr{R}(F_{B_1}^*) = \mathscr{N}(B_1). \tag{3.5}$$

Since $\mathscr{R}(G^\dagger) = \mathscr{R}(G^*)$ and $\mathscr{R}[(G^\dagger)^*] = \mathscr{R}(G)$, the two conditions in (3.5) are equivalent to $GC_1 = 0$ and $B_1 G = 0$, namely, $GCA^\dagger A = GC$ and $AA^\dagger BG = BG$, which are equivalent to (3.3). $\square$

Note that the equation in (1.6) is in fact equivalent to the quadratic matrix equation

$$\begin{bmatrix} A & B \\ C & ? \end{bmatrix}\begin{bmatrix} ? & ? \\ ? & G \end{bmatrix} = I_t. \tag{3.6}$$

It is obvious that (3.6) is solvable if and only if

$$\min_{?} r\left(I_t - \begin{bmatrix} A & B \\ C & ? \end{bmatrix}\begin{bmatrix} ? & ? \\ ? & G \end{bmatrix} \right) = 0, \tag{3.7}$$

or equivalently,

$$\min_{?} r\begin{bmatrix} ? & ? & I_n & 0 \\ ? & G & 0 & I_k \\ I_m & 0 & A & B \\ 0 & I_l & C & ? \end{bmatrix} = t. \tag{3.8}$$

It is not difficult to derive the minimal rank on the left-hand side of (3.8) from (1.18) and a formula for the minimal rank of $A - B_1X_1C_1 - B_2X_2C_2$ given in [30]. Hence (3.1), (3.2) and (3.3) can also be derived from (3.8). The details are omitted.

The second equation in (3.4) shows that the matrix X satisfying (1.6) is nothing but the solution of a linear matrix equation composed of A, B, C and G. Solving the equation for the unknown matrix X yields the following result.

Theorem 3.2. *Let $M(X)$ be as given in (1.5). Under the conditions in (3.1), (3.2) and (3.3), the general solution to (1.6) can be written in the following parametric form*

$$X = G^\dagger + CA^\dagger B + CF_A V + WE_A B, \tag{3.9}$$

where $V \in \mathbb{C}^{n\times k}$ and $W \in \mathbb{C}^{l\times m}$ are arbitrary.

Combining Theorems 3.1 and 3.2 with (1.4) yields the following result.

Theorem 3.3. *Let $M(X)$ be as given in (1.5). Then,*

(a) *There exists a matrix X such that (1.6) holds if and only if A, B, C and G satisfy the five conditions in (3.1), (3.2) and (3.3).*
(b) *Under (3.1), (3.2) and (3.3), the general solution to (1.6) can be written as (3.9).*
(c) *Under (3.1), (3.2), (3.3) and (3.9), the inverse of $M(X)$ can be expressed as*

$$M^{-1}(X) = \begin{bmatrix} I_n & -A^{\dagger}B \\ 0 & I_k \end{bmatrix} \begin{bmatrix} A^{\dagger} + C_1^{\dagger}(SGS - S)B_1^{\dagger} & C_1^{\dagger} - C_1^{\dagger}SG \\ B_1^{\dagger} - GSB_1^{\dagger} & G \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -CA^{\dagger} & I_l \end{bmatrix}$$
$$= \begin{bmatrix} A^{\dagger} + A^{\dagger}BGCA^{\dagger} + T & C_1^{\dagger} - A^{\dagger}BG - C_1^{\dagger}C_1VG \\ B_1^{\dagger} - GCA^{\dagger} - GWB_1B_1^{\dagger} & G \end{bmatrix}, \tag{3.10}$$

where $B_1 = E_AB$, $C_1 = CF_A$, $S = G^{\dagger} + C_1V + WB_1$, and

$$T = C_1^{\dagger}C_1VGWB_1B_1^{\dagger} - C_1^{\dagger}C_1VB_1^{\dagger} - C_1^{\dagger}WB_1B_1^{\dagger} - A^{\dagger}B(I_k - GWB_1)B_1^{\dagger}$$
$$- C_1^{\dagger}(I_l - C_1VG)CA^{\dagger}.$$

(d) *If $V = 0$ and $W = 0$ in (3.10), then*

$$M^{-1}(X) = \begin{bmatrix} A^{\dagger} + A^{\dagger}BGCA^{\dagger} - A^{\dagger}BB_1^{\dagger} - C_1^{\dagger}CA^{\dagger} & C_1^{\dagger} - A^{\dagger}BG \\ B_1^{\dagger} - GCA^{\dagger} & G \end{bmatrix}. \tag{3.11}$$

Three special cases of Theorem 3.3 are given below.

Corollary 3.4. *Let $M(X)$ be as given in (1.5), and suppose that the matrices A, B and C in (1.5) satisfy*

$$\mathscr{R}(A) \cap \mathscr{R}(B) = \{0\} \quad \text{and} \quad \mathscr{R}(A^*) \cap \mathscr{R}(C^*) = \{0\}. \tag{3.12}$$

Then there exists a matrix X such that (1.6) holds if and only if the following five conditions

$$r(A) + r(B) = m, \qquad r(A) + r(C) = n, \qquad r(G) = t - m - n + r(A), \tag{3.13}$$
$$BG = 0, \qquad GC = 0 \tag{3.14}$$

hold. In this case, the general solution to (1.6) can be written as

$$X = G^{\dagger} + CA^{\dagger}B + CV + WB, \tag{3.15}$$

where $V \in \mathbb{C}^{n\times k}$ and $W \in \mathbb{C}^{l\times m}$ are arbitrary, and $M^{-1}(X)$ is given by

$$M^{-1}(X) = \begin{bmatrix} A^{\dagger} - A^{\dagger}BB_1^{\dagger} - C_1^{\dagger}CA^{\dagger} + T & C_1^{\dagger} - C_1^{\dagger}CVG \\ B_1^{\dagger} - GWB_1^{\dagger} & G \end{bmatrix}, \tag{3.16}$$

where $B_1 = E_AB$, $C_1 = CF_A$, and $T = C_1^{\dagger}CVGWBB_1^{\dagger} - C_1^{\dagger}CVB_1^{\dagger} - C_1^{\dagger}WBB_1^{\dagger}$.

Proof. Under (3.12), the five conditions in (3.1), (3.2) and (3.3) reduce to (3.13) and (3.14). In this case,

$$\mathscr{R}(C) = \mathscr{R}(CF_A), \qquad \mathscr{R}(B^*) = \mathscr{R}[(E_AB)^*], \qquad (CF_A)(CF_A)^{\dagger} = CC^{\dagger}, \qquad (E_AB)^{\dagger}(E_AB) = B^{\dagger}B.$$

Thus, (3.9) reduces to (3.10) and (3.15) reduces to (3.16). $\square$

Corollary 3.5. *Let $M(X)$ be as given in (1.5). Then there exists a matrix X such that*

$$M^{-1}(X) = \begin{bmatrix} A & B \\ C & X \end{bmatrix}^{-1} = \begin{bmatrix} ? & ? \\ ? & 0 \end{bmatrix} \tag{3.17}$$

holds if and only if the following five conditions

$$\mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}, \qquad \mathscr{R}(A^*) \cap \mathscr{R}(C^*) = \{0\}, \tag{3.18}$$
$$r(A) = m + n - t, \qquad r(B) = k, \qquad r(C) = l \tag{3.19}$$

hold. In this case, the matrix X satisfying (3.17) can be chosen arbitrarily, and $M^{-1}(X)$ is given by

$$M^{-1}(X) = \begin{bmatrix} A^{\dagger} - A^{\dagger}BB_1^{\dagger} - C_1^{\dagger}CA^{\dagger} - C_1^{\dagger}(X - CA^{\dagger}B)B_1^{\dagger} & C_1^{\dagger} \\ B_1^{\dagger} & 0 \end{bmatrix}, \tag{3.20}$$

where $B_1 = E_AB$ and $C_1 = CF_A$.

Proof. The five conditions in (3.1), (3.2) and (3.3) reduce to (3.18) and (3.19) if $G = 0$ in (1.6). In this case, the inverse of $M(X)$ in (3.17) can be written as (3.20) from Corollary 2.2. $\square$

Corollary 3.6. *Let $M(X)$ be as given in (1.5) with $A = 0$. Then there exists a matrix X such that*

$$M^{-1}(X) = \begin{bmatrix} 0 & B \\ C & X \end{bmatrix}^{-1} = \begin{bmatrix} ? & ? \\ ? & G \end{bmatrix} \tag{3.21}$$

holds if and only if the following five conditions

$$r(B) = m, \qquad r(C) = n, \qquad r(G) = t - m - n, \qquad BG = 0, \qquad GC = 0$$

hold. In this case, the general solution of X to (3.21) can be written as

$$X = G^{\dagger} + CV + WB,$$

where $V \in \mathbb{C}^{n\times k}$ and $W \in \mathbb{C}^{l\times m}$ are arbitrary, and $M^{-1}(X)$ is given by

$$M^{-1}(X) = \begin{bmatrix} C^{\dagger}CVGWBB^{\dagger} - C^{\dagger}CVB^{\dagger} - C^{\dagger}WBB^{\dagger} & C^{\dagger} - C^{\dagger}CVG \\ B^{\dagger} - GWBB^{\dagger} & G \end{bmatrix}.$$

Proof. It follows from Theorem 3.3 by setting $A = 0$. $\square$

Solutions to the completion problems corresponding to (1.8) and (1.10) are given in the following two theorems.

Theorem 3.7. *Let $M_1(X)$ be as given in (1.7). Then,*

(a) *There exists an Hermitian matrix X such that (1.8) holds if and only if the following three conditions*

$$r[A \mid B] = m, \qquad r(A) - r(G) = m - n, \qquad \mathscr{R}(BG) \subseteq \mathscr{R}(A) \tag{3.22}$$

hold.

(b) *Under (3.22), the general Hermitian solution to (1.8) can be written in the parametric form*

$$X = G^{\dagger} + B^{*}A^{\dagger}B + (E_AB)^{*}W^{*} + WE_AB, \tag{3.23}$$

where $W \in \mathbb{C}^{n\times m}$ is arbitrary.

(c) *Under (3.22) and (3.23), the inverse of $M_1(X)$ is given by*

$$\begin{aligned} M_1^{-1}(X) &= \begin{bmatrix} I_m & -A^{\dagger}B \\ 0 & I_n \end{bmatrix} \begin{bmatrix} A^{\dagger} + (B_1^{*})^{\dagger}(SGS - S)B_1^{\dagger} & (B_1^{*})^{\dagger} - (B_1^{*})^{\dagger}SG \\ B_1^{\dagger} - GSB_1^{\dagger} & G \end{bmatrix} \begin{bmatrix} I_m & 0 \\ -B^{*}A^{\dagger} & I_n \end{bmatrix} \\ &= \begin{bmatrix} A^{\dagger} + A^{\dagger}BGB^{*}A^{\dagger} + T & (B_1^{*})^{\dagger} - A^{\dagger}BG - B_1B_1^{\dagger}W^{*}G \\ B_1^{\dagger} - GB^{*}A^{\dagger} - GWB_1B_1^{\dagger} & G \end{bmatrix}, \end{aligned} \tag{3.24}$$

where $B_1 = E_AB$ and $S = G^{\dagger} + B_1^{*}W^{*} + WB_1$, and

$$\begin{aligned} T \;=\; & B_1B_1^{\dagger}W^{*}GWB_1B_1^{\dagger} - B_1B_1^{\dagger}W^{*}B_1^{\dagger} - (B_1^{*})^{\dagger}WB_1B_1^{\dagger} - A^{\dagger}B(I_n - GWB_1)B_1^{\dagger} \\ & - (B_1^{*})^{\dagger}[I_n - (B_1^{*})W^{*}G]B^{*}A^{\dagger}. \end{aligned}$$

(d) *If $W = 0$ in (3.23), then*

$$M_1^{-1}(X) = \begin{bmatrix} A^{\dagger} + A^{\dagger}BGB^{*}A^{\dagger} - A^{\dagger}BB_1^{\dagger} - (B_1^{*})^{\dagger}B^{*}A^{\dagger} & (B_1^{*})^{\dagger} - A^{\dagger}BG \\ B_1^{\dagger} - GB^{*}A^{\dagger} & G \end{bmatrix}. \tag{3.25}$$

Proof. It can be seen from (1.4) and Lemma 1.3(b) and (c) that (1.8) has an Hermitian solution for X if and only if both A and B satisfy

$$r\begin{bmatrix} A \\ B^{*} \end{bmatrix} = m \quad \text{and} \quad r[A \mid B] = m, \tag{3.26}$$

and there exists a matrix X satisfying the following three conditions

$$r[F_{B_1}(X - B^{*}A^{\dagger}B)F_{B_1}] = t - 2m + r(A), \tag{3.27}$$

$$[F_{B_1}(X - B^{*}A^{\dagger}B)F_{B_1}]^{\dagger} = G, \quad X = X^{*}. \tag{3.28}$$

The two rank equalities in (3.26) are the same, which is the first rank equality in (3.22), while (3.27) and (3.28) are equivalent to

$$r(G) = t - 2m + r(A) = n - m + r(A), \tag{3.29}$$

$$F_{B_1}(X - B^*A^\dagger B)F_{B_1} = G^\dagger, \quad X = X^*. \tag{3.30}$$

Eq. (3.29) reduces to the second equality in (3.22). It can also be derived from Lemma 1.5 that the left-hand side equation in (3.30) has an Hermitian solution for X if and only if $\mathscr{R}(G^\dagger) \subseteq \mathscr{R}(F_{B_1})$, which is equivalent to the third range inclusion in (3.22), and the general solution of (3.30) under (3.22) can be written as (3.23). In this case, the inverse of $M_1(X)$ can be expressed as (3.24) by (1.4). □

In the same manner, we can also derive a complete solution to (1.10) from (1.4) and Lemma 1.6.

Theorem 3.8. *Let $M_2(X)$ be as given in (1.9). Then,*

(a) *There exists a skew-Hermitian matrix X such that (1.10) holds if and only if*

$$r[A \mid B] = m, \qquad r(A) - r(G) = m - n \quad \text{and} \quad \mathscr{R}(BG) \subseteq \mathscr{R}(A) \tag{3.31}$$

hold.

(b) *Under (3.31), the general skew-Hermitian solution to (1.10) can be written in the parametric form*

$$X = G^\dagger - B^*A^\dagger B - (E_AB)^*W^* + WE_AB, \tag{3.32}$$

where $W \in \mathbb{C}^{n\times m}$ is arbitrary.

(c) *Under (3.31) and (3.32), the inverse of $M_2(X)$ can be expressed as*

$$\begin{aligned} M_2^{-1}(X) &= \begin{bmatrix} I_m & -A^\dagger B \\ 0 & I_n \end{bmatrix} \begin{bmatrix} A^\dagger - (B_1^*)^\dagger(SGS - S)B_1^\dagger & -(B_1^*)^\dagger - (B_1^*)^\dagger SG \\ B_1^\dagger - GSB_1^\dagger & G \end{bmatrix} \begin{bmatrix} I_m & 0 \\ B^*A^\dagger & I_n \end{bmatrix} \\ &= \begin{bmatrix} A^\dagger - A^\dagger BGB^*A^\dagger + T & -(B_1^*)^\dagger - A^\dagger BG + B_1B_1^\dagger W^*G \\ B_1^\dagger + GB^*A^\dagger - GWB_1B_1^\dagger & G \end{bmatrix}, \end{aligned}$$

where $B_1 = E_AB$, $S = G^\dagger - B_1^*W^* + WB_1$, and

$$T = B_1B_1^\dagger W^*GWB_1B_1^\dagger - B_1B_1^\dagger W^*B_1^\dagger + (B_1^*)^\dagger WB_1B_1^\dagger - A^\dagger B(I_n - GWB_1)B_1^\dagger - (B_1^*)^\dagger[I_n + (B_1^*)W^*G]B^*A^\dagger.$$

(d) *If $W = 0$ in (3.32), then*

$$M_2^{-1}(X) = \begin{bmatrix} A^\dagger - A^\dagger BGB^*A^\dagger - A^\dagger BB_1^\dagger - (B_1^*)^\dagger B^*A^\dagger & -(B_1^*)^\dagger - A^\dagger BG \\ B_1^\dagger + GB^*A^\dagger & G \end{bmatrix}.$$

4. Concluding remarks

In the previous sections, we gave a formula for the inverse of any 2×2 nonsingular partitioned matrix through the Moore–Penrose inverses of the submatrices in the partitioned matrix. Based on this formula, we derived general solutions to three inverse completion problems.

As generalizations of the work on matrix inverse completion problems for 2×2 partitioned matrices, we propose some other inverse completion problems for further consideration:

(a) Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, $G_1 \in \mathbb{C}^{n\times m}$, $G_2 \in \mathbb{C}^{n\times l}$ and $G_3 \in \mathbb{C}^{k\times m}$ be given with $m+l = n+k = t$. Find a matrix $X \in \mathbb{C}^{l\times k}$ such that

$$\begin{bmatrix} A & B \\ C & X \end{bmatrix}^{-1} = \begin{bmatrix} G_1 & ? \\ ? & ? \end{bmatrix}, \qquad \begin{bmatrix} A & B \\ C & X \end{bmatrix}^{-1} = \begin{bmatrix} ? & G_2 \\ ? & ? \end{bmatrix}, \qquad \begin{bmatrix} A & B \\ C & X \end{bmatrix}^{-1} = \begin{bmatrix} ? & ? \\ G_3 & ? \end{bmatrix} \tag{4.1}$$

hold, respectively.

(b) Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{l\times k}$, $C \in \mathbb{C}^{n\times l}$ and $D \in \mathbb{C}^{k\times m}$ be given with $m+l = n+k = t$. Find two matrices $X \in \mathbb{C}^{m\times k}$ and $Y \in \mathbb{C}^{l\times n}$ such that

$$\begin{bmatrix} A & X \\ Y & B \end{bmatrix}^{-1} = \begin{bmatrix} ? & C \\ D & ? \end{bmatrix} \tag{4.2}$$

holds.

(c) Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{l\times k}$, $C \in \mathbb{C}^{n\times m}$ and $D \in \mathbb{C}^{k\times l}$ be given with $m + l = n + k = t$. Find two matrices $X \in \mathbb{C}^{m\times k}$ and $Y \in \mathbb{C}^{l\times n}$ such that

$$\begin{bmatrix} A & X \\ Y & B \end{bmatrix}^{-1} = \begin{bmatrix} C & ? \\ ? & D \end{bmatrix} \tag{4.3}$$

holds.

(d) Let $A = \pm A^* \in \mathbb{C}^{m\times m}$, $B = \pm B^* \in \mathbb{C}^{k\times k}$ and $C \in \mathbb{C}^{m\times k}$ be given. Find a matrix $X \in \mathbb{C}^{m\times k}$ such that

$$\begin{bmatrix} A & X \\ \pm X^* & B \end{bmatrix}^{-1} = \begin{bmatrix} ? & C \\ \pm C^* & ? \end{bmatrix} \tag{4.4}$$

holds.

(e) Let $A = \pm A^* \in \mathbb{C}^{m\times m}$, $B = \pm B^* \in \mathbb{C}^{k\times k}$, $C = \pm C^* \in \mathbb{C}^{m\times m}$ and $D = \pm D^* \in \mathbb{C}^{k\times k}$ be given. Find a matrix $X \in \mathbb{C}^{m\times k}$ such that

$$\begin{bmatrix} A & X \\ \pm X^* & B \end{bmatrix}^{-1} = \begin{bmatrix} C & ? \\ ? & D \end{bmatrix} \tag{4.5}$$

holds.

(f) Let $A, B \in \mathbb{C}^{m\times k}$ be given. Find two matrices $X = \pm X^* \in \mathbb{C}^{m\times m}$ and $Y = \pm Y^* \in \mathbb{C}^{k\times k}$ such that

$$\begin{bmatrix} X & A \\ \pm A^* & Y \end{bmatrix}^{-1} = \begin{bmatrix} ? & B \\ \pm B^* & ? \end{bmatrix} \tag{4.6}$$

holds.

Analogous to (3.6) and (3.7), the above matrix inverse completion problems are equivalent to certain matrix rank completion problems. For instance, a necessary condition for (4.2) to hold is

$$\max_{?} r \begin{bmatrix} A & ? \\ ? & B \end{bmatrix} = \max_{?} r \begin{bmatrix} ? & C \\ D & ? \end{bmatrix} = t,$$

while a necessary and sufficient condition for (4.2) to hold is

$$\min_{?} r \left(\begin{bmatrix} A & ? \\ ? & B \end{bmatrix} \begin{bmatrix} ? & C \\ D & ? \end{bmatrix} - I_t \right) = 0.$$

It is natural to replace the inverses of the matrices in (1.6), (1.8) and (1.10), as well as in (4.1)–(4.6) with generalized inverses of matrices. In such cases, it would be of interest to consider the corresponding completion problems for generalized inverses of partial matrices. A more challenging task in this area is to consider various invertible completions of 2×2 partial operator matrices; see, e.g., [9,13,19,23]. In this case, Moore–Penrose inverses of operators will play the same roles as those of matrices. However, the matrix rank method is no longer available to characterize the existence of inverse completions of partial operator matrices. Instead, ranges, kernels, spectra, norms, dimensions and index of operators can be used to tackle these completion problems, but the results obtained seem quite messy.

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